Considerations on the three-body problem

Leonhard Euler

The problem of determining the motion of three bodies which mutually attract each other, according to the Newtonian hypothesis, has for some time become so famous by the cares that the greatest Geometers have used on it, that we have already started to argue about to whom the glory of having the first solution belonged. But this dispute is very premature, and we are still quite a long way from reaching a perfect solution to this problem. All that has been done with it until now is restricted to a very particular case, where the motion of each of the three bodies follows approximately the rules established by *Kepler*, and even in this case we are limited to determining the motion by approximation. In all the other cases, we cannot brag that one may determine only approximately the motion of the three bodies, which still remains for us as great a mystery as if one had never thought about this problem.

§2. To clearly prove how far we still are from a complete solution to this problem, we need only compare it with the case where there are only two bodies which mutually attract each other, and even with the simplest case, where it is a matter of determining the motion of a body projected in an arbitrary way into the void. And one will easily agree that it would have been impossible to find the parabola which such a body describes, without having known beforehand the law in which a heavy body falls perpendicularly lower. Without the discovery of Galileo, that the speed of a falling body increases in proportion to the square root of the height, we would certainly never have attained the knowledge of the parabola that a body thrown obliquely into the void describes.

§3. It is the same for the general motion of two bodies under mutual attraction, where it is also necessary to start by determining the rectilinear motion (in which the bodies are approaching each other or pulling away from each other) before we may undertake the search for the conic sections which these bodies describe when they are thrown obliquely. For, although the great Newton followed a reverse order in his investigations, nobody could doubt that he would never have succeeded in determining the curvilinear motion, without having been in a position to determine the rectilinear.

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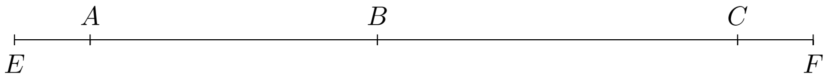
§4. From this I draw this incontestable conclusion, that we cannot hope to solve the general three body problem, unless we have found a way to solve the case where the three bodies move on a straight line. This happens when they have been initially arranged on a straight line, and when they have been either at rest or pushed along the same direction. So, before undertaking the solution of the three body problem as it is commonly stated, it is indispensably necessary to apply oneself to the case where the motion of all three bodies is done on the same straight line. We can be well assured that, as long as this last problem eludes our investigations, we will try in vain to succeed in the solution of the first. In such difficult investigations, it is always appropriate to start with the simplest cases.

§5. The case where the three bodies move on the same straight line is admittedly much simpler than if these bodies described curved lines, where it could even happen that these curves are not found in the same plane. These details must necessarily make our investigations much more complicated. This is so evident, that we will be very surprised that none of the great Geometers who have occupied themselves with this problem have started their investigations with the case of rectilinear motion. But the reason undoubtedly is that such motion is not found in the world, and that these great men were a little impatient to apply the result of their works to real motions in the heavens, without wanting to undertake investigations which would not have an immediate application there.

§6. Perhaps one will even be tempted to believe that this case, because of its simplicity, was too far below the abilities of these Geometers, and that they wanted to leave the development to lesser geniuses. But this would be a very unfounded opinion, since the solution to this case is subject to such great difficulties that they seem not to have yet been able to be surmounted by the greatest Analysts. So it seems to me very important to put all these difficulties in plain sight, in order that those who will yet want to occupy themselves with this great three-body problem may gather their forces in order to surmount the difficulties, if possible. These efforts will be all the more useful because we cannot hope to ever reach a perfect solution to this problem unless a way has previously been found to overcome all the difficulties surrounding the rectilinear case; but even then, perhaps we would not be much better off with regard to the general problem.

§7. So let three bodies be moving along the straight line EF, and be presently found at points A, B, C, the letters A, B, C being taken at the same time to indicate their respective masses. Then, setting the distances $AB = x$



and $BC = y$, the body A will be pushed towards F by the accelerative forces

$$\frac{B}{xx} + \frac{C}{(x+y)^2} \,,$$

the body B will be pushed in the same direction towards F by the accelerative force

$$\frac{C}{yy} - \frac{A}{xx} \,,$$

and the body C towards E by the force

$$\frac{B}{yy} + \frac{A}{(x+y)^2} \,.$$

Let us consider the body B as at rest, or better let us seek the respective motions of the two others A and C in relation to B; and since we must transform the forces which are acting on B towards the others, by reversing their directions, the body A will be pushed towards B by the force

$$\frac{A+B}{xx} - \frac{C}{yy} + \frac{C}{(x+y)^2} \,,$$

and the body C towards B by the force

$$\frac{B+C}{yy} - \frac{A}{xx} + \frac{A}{(x+y)^2} \,.$$

§8. Now let us suppose the element of time is $\mathrm{d}t$, taking it to be constant, and the principles of mechanics immediately give us these two equations:

$$\text{I.} \quad \frac{\mathrm{d}\,\mathrm{d}x}{\mathrm{d}t^2} = \frac{-A-B}{xx} + \frac{C}{yy} - \frac{C}{(x+y)^2}$$

$$\text{II.} \quad \frac{\mathrm{d}\,\mathrm{d}y}{\mathrm{d}t^2} = \frac{-B-C}{yy} + \frac{A}{xx} - \frac{A}{(x+y)^2}$$

where I do not bother with the coefficient that would have to be given to the element $\mathrm{d}t$, which depends on the way we want to express time. And so determining the motion of the bodies A and C, in relation to the body B, depends solely on the resolution of these two equations, so that the problem is reduced to a purely analytic question.

§9. *Particular case.* Before undertaking the resolution of these equations, I observe that there is a case where all the difficulties vanish; for it is easy to see that a case would be possible where the distances x and y are always maintaining the same ratio. In order to find this case, set $y = nx$, and we will have

$$-n(A+B) + \frac{C}{n} - \frac{Cn}{(1+n)^2} = \frac{-B-C}{nn} + A - \frac{A}{(n+1)^2}\,,$$

or as well

$$n^3(nn+3n+3)A + (n^5+2n^4+n^3-nn-2n-1)B \\ -(3nn+3n+1)C = 0,$$

from which it is easy to find the correct ratio among the masses A, B, C when the number n is given, in order that this case may occur. But if the masses are given, in order to find the number n we must solve this fifth degree equation:

$$(A+B)n^5 + (3A+2B)n^4 + (3A+B)n^3 - (B+3C)nn \\ -(2B+3C)n - B - C = 0,$$

and then, setting

$$A + B - \frac{C}{nn} + \frac{C}{(1+n)^2} = E,$$

we will have for the motion

$$\frac{\mathrm{d}x^2}{2\,\mathrm{d}t^2} = E\left(\frac{1}{x} - \frac{1}{a}\right)$$

and

$$\mathrm{d}t\sqrt{2E} = \frac{-\,\mathrm{d}x\sqrt{ax}}{\sqrt{a-x}}.$$

§10. Having then found the right value for the number n, so that we always have $y = nx$, this case will happen when the distances BA and BC are initially as 1 to n, and the speeds imparted then towards B will have the same ratio. Then the motion of the body A towards B will be the same as that of an infinitely small body towards a body whose mass would be equal to E. To best determine this motion, we only have to put $x = a\cos^2\theta$, in order to have

$$\mathrm{d}t\sqrt{2}E = 2a^{\frac{3}{2}}\,\mathrm{d}\theta\cos^2\theta,$$

and therefore

$$t\sqrt{2}E = a^{\frac{3}{2}}(\theta + \sin\theta\cos\theta)$$

where a indicates the distance AB at the beginning, when we had $t = 0$ and $\theta = 0$, by supposing that the body A is then found at rest. It will then arrive at B, when making $\theta = 90^\circ = \pi/2$, after the time t determined by this equality:

$$t\sqrt{2}E = a^{\frac{3}{2}}\frac{\pi}{2}$$

This way of representing the motion, by introducing the arcs of the circle into it, seems to be most appropriate to this design.